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Note

The Unification of Certain Enumeration Problems for Sequences

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A unified treatment is presented for certain enumeration problems involving sequences over a finite set. The problems concern permutations, derangements, partitions and compositions, and include the derangement, Smirnov and Simon Newcomb problems and their various generalizations. A single theorem includes all of these problems as special cases.

1. PRELIMINARIES AND NOTATION

Let $\mathscr{L}_m = \{1, 2,..., m\}$. Then the sequence $\sigma = (\sigma_1, \sigma_2,..., \sigma_n) \in \mathscr{S} \equiv \bigcup_{n=1}^{\infty} \mathscr{L}_m{}^n$ has *length* $\lambda(\sigma) = n$, and *type* $\tau(\sigma) = \mathbf{b} = (b_1,..., b_m)$ where σ has b_i i's, $i = 1, 2,..., m$. Let $\tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2,..., \tilde{\sigma}_n)_{\leqslant}$ be the sequence obtained by rearranging the elements of σ in nondecreasing order. Thus $\tau(\tilde{\sigma}) = \tau(\sigma)$. Let u, x, y, z, $\mathbf{y} = (y_1,..., y_m)$ and $\mathbf{z} = (z_1,..., z_m)$ be indeterminates, and for notational convenience write $b_1! \cdots b_m!$ as $\mathbf{b}!$ and $y_1^{b_1} \cdots y_m^{b_m}$ as $\mathbf{y}^{\mathbf{b}}$, and let $\mathbf{1}$ be the vector of m 1's. The functions $\omega(\sigma)$ and $\omega^*(\sigma)$ are defined by

$$\omega(\sigma) = xy \prod_{i=2}^{\lambda(\sigma)} \xi(\sigma_i, \sigma_{i-1}), \qquad \omega^*(\sigma) = \prod_{i=1}^{\lambda(\sigma)} \xi(\sigma_i, \tilde{\sigma}_i),$$

where

$$\begin{aligned} \xi(i,j) &= x \quad \text{if} \quad i < j, \\ &= y \quad \text{if} \quad i > j \quad \text{where} \quad i, j \in \mathscr{L}_m, \\ &= z \quad \text{if} \quad i = j. \end{aligned}$$

We shall call (σ_{i-1}, σ_i) a *fall*, *rise*, or *level* in σ in the cases $\sigma_i < \sigma_{i-1}$,

$\sigma_i > \sigma_{i-1}$, and $\sigma_i = \sigma_{i-1}$ respectively. Let Ψ_m be the generating function of the set $\mathscr{S}$ with respect to the weight function ω. Thus

$$\Psi_m = \sum_{\sigma \in \mathscr{S}} \omega(\sigma)(u\mathbf{y})^{\tau(\sigma)}.$$

$\Psi_m{}^*$ is defined with ω^* replacing ω, and $[u^n\mathbf{y}^\mathbf{b}]\,\Psi_m$ denotes the coefficient of $u^n\mathbf{y}^\mathbf{b}$ in Ψ_m. A relationship between two generating functions is given in Section 2, and explicit expressions for them are given in Section 3. Section 4 shows that the solutions of a number of well-known problems are in fact straightforward specializations of these expressions.

2. Combinatorial Theorems

We begin by constructing a matrix A whose permanent has particular significance in the enumeration of the elements of $\mathscr{S}$.

Definition 2.1. For the ordered partitions $\mathbf{a}$, $\mathbf{b}$ of some integer $n \geqslant 1$ into m nonempty parts, $A(\mathbf{b}, \mathbf{a})$ is the $n \times n$ matrix A such that $(A[\rho_i \mid \kappa_j])_{pq} = \xi(i, j)$ where $\rho_i = \{j \mid b_1 + \cdots + b_{i-1} + 1 \leqslant j \leqslant b_1 + \cdots + b_i\}$, $\kappa_i = \{j \mid a_1 + \cdots + a_{i-1} + 1 \leqslant j \leqslant a_1 + \cdots + a_i\}$, $i = 1,\ldots, m$ and $A[\sigma_i \mid \kappa_j]$ is the submatrix of A intercepted by the rows and columns of A with labels in ρ_i and κ_j respectively. □

Lemma 2.1. $\Psi_m{}^* = \sum_{\mathbf{b} \geqslant 1} \operatorname{per} A(\mathbf{b}, \mathbf{b})(u\mathbf{y})^\mathbf{b}(\mathbf{b}!)^{-1}$.

Proof. $\operatorname{per}(A) = \sum_{\pi \in \mathscr{S}_n} a_{i,\pi(1)} \cdots a_{n,\pi(n)}$ where $n = b_1 + \cdots + b_m$ and $A = [a_{ij}] = A(\mathbf{b}, \mathbf{b})$. Since A encodes ω^*, $\operatorname{per}(A)$ enumerates the sequences, assuming all elements distinguishable. But elements of the same type are indistinguishable. Thus $|\operatorname{aut}(\sigma)| = \mathbf{b}!$ where $\sigma \in \mathscr{L}_m{}^n$ and $\tau(\sigma) = \mathbf{b}$. The lemma follows. □

The following lemma obtains Ψ_{m-1} in terms of $\Psi_m{}^*$ by representing the linear sequences of $\mathscr{S}$ as rooted cycles, and by considering graphs each of whose components are cycles.

Lemma 2.2. $\Psi_{m-1} = uz + u^{-1}\mathbf{L}_{y_m}(\partial/\partial y_m) \log \Psi_m{}^*$ *where* $\mathbf{L}_{y_m}(\cdots) = (\cdots)\,|_{y_m=0}$.

Proof. Let $\sigma \in \mathscr{L}_m{}^n$, $\tau(\sigma) = \mathbf{b}$ and consider an element of σ labeled m. This element exists but may not be unique since $b_m \geqslant 1$. Suppose it is unique and root the sequence at this element. Thus $\sigma - \{m\} \in \mathscr{L}_{m-1}^{n-1}$ and the construction is bijective between the sequences of $\mathscr{L}_m{}^n$ rooted on a unique m, and $\mathscr{X}_{m-1}^{n-1}$. The sequences are enumerated by $\Psi_{m-1} - uz$. The term uz enumerates a 1-cycle and is excluded because the construction would yield the null sequence $(\varnothing)$ and $\tau(\varnothing) \ngeqslant \mathbf{1}$. It remains to force $b_m = 1$ so that there is a unique vertex labeled m. Now $\log \Psi_m{}^*$ is the counting series for directed, labeled circuits whose vertices are labeled, not necessarily uniquely. A rooted, directed labeled cycle represents a linear sequence. Thus

$$u^{-1}\mathbf{L}_{y_m}(\partial/\partial y_m) \log \Psi_m{}^* = \sum_{n=1}^{\infty} \sum_{\sigma \in \mathscr{X}_m^n} u^{-1}\omega(\sigma)\, u^{\lambda(\sigma)} y_1^{b_1} \cdots y_{m-1}^{b_{m-1}} \delta_{b_m - 1, 0}$$

$$= \sum_{n=2}^{\infty} \sum_{\sigma \in \mathscr{X}_{m-1}^{n-1}} \omega(\sigma)(u\mathbf{y})^{\mathbf{b}} = \Psi_{m-1} - uz,$$

and the result follows. □

3. Counting Series

Theorem 3.1 (Main theorem). *Let $D = A(\mathbf{1}, \mathbf{1})$. Then*

$$1 + \sum_{\mathbf{a},\mathbf{b} \geqslant \mathbf{1}} \operatorname{per}(A(\mathbf{b}, \mathbf{a})) \frac{\mathbf{y}^{\mathbf{b}}\mathbf{z}^{\mathbf{a}}}{\mathbf{a}!\mathbf{b}!} = \exp(\mathbf{y}\, D\, \mathbf{z}^T).$$

Proof. Let $(B_1, \ldots, B_m)$ denote the blocks of the row partition of $A(\mathbf{b}, \mathbf{a})$, and $\theta(j)$ be the number of the first row of B_j. The m-fold Laplace expansion for the permanent of $A = A(\mathbf{b}, \mathbf{a})$ gives

$$\operatorname{per}(A) = \sum \prod_{j=1}^{m} \operatorname{per}(A[B_j \mid \alpha_j]) = \mathbf{b}! \sum_{j=1}^{m} \left[\prod_{e \in A[\theta(j) \mid \kappa_j]} e \right]$$

$$= \mathbf{b}![\mathbf{x}^{\mathbf{1}}\mathbf{y}^{\mathbf{b}}] \prod_{i,j=1}^{m} (1 + A[\theta(i) \mid j]\, x_j y_i),$$

where the summations are over $\alpha_1, \ldots, \alpha_m$ such that $\bigcup_{i=1}^{m} \alpha_i = \mathscr{X}_n$, and $|\alpha_j| = b_j$ for $j = 1, \ldots, m$. These conditions are imposed by the operators $[\mathbf{x}^{\mathbf{1}}]$ and $[\mathbf{y}^{\mathbf{b}}]$. Thus, letting $D = [d_{ij}]$ we have

$$(1/\mathbf{b}!) \operatorname{per}(A) = [\mathbf{y}^{\mathbf{b}}] \prod_{j=1}^{n} \sum_{i=1}^{m} A[\theta(i) \mid j]\, y_i = [\mathbf{y}^{\mathbf{b}}] \prod_{j=1}^{m} \left(\sum_{i=1}^{m} d_{ij} y_i \right)^{a_j},$$

and the theorem follows. □

LEMMA 3.1. *Let $D_j = A(\mathbf{1}, \mathbf{1})$. Then*

$$| D_j | = \frac{x(z-y)^j - y(z-x)^j}{x-y}$$

where D is $j \times j$.

Proof. Straightforward. □

THEOREM 3.2.

$$\Psi_m{}^* = \frac{x-y}{x \prod_{j=1}^m \{1 + (y-z)\, uy_j\} - y \prod_{j=1}^m \{1 + (x-z) uy_j\}}.$$

Proof. From Theorem 3.1 and the MacMahon Master Theorem (MacMahon [7], Cartier and Foata [4]) we have:

$$\operatorname{per}(A)/\mathbf{b}! = [\mathbf{y}^{\mathbf{b}}] \prod_{j=1}^{m} \left(\sum_{i=1}^{m} d_{ij} \right)^{b_j} \quad \text{and} \quad \sum_{\mathbf{b}} (\operatorname{per}(A)/\mathbf{b}!)\, \mathbf{y}^{\mathbf{b}} = | I - YD |^{-1},$$

where $A \equiv A(\mathbf{b}, \mathbf{b})$, $Y = \operatorname{diag}(y_1, ..., y_m)$ and I is the $m \times m$ identity matrix. Taking the determinant of the sum (Marcus [8]) we have:

$$| I - YD | = \sum_{j=0}^{m} (-1)^j \sum_{\alpha} |(YD)[\alpha \mid \alpha]| = \sum_{j=0}^{m} (-1)^j \sum_{\alpha} (y_{\alpha_1} \cdots y_{\alpha_j}) \mid D_j \mid,$$

where the summation is over $\alpha \subseteq \mathscr{L}_m$ such that $| \alpha | = j$. The theorem follows immediately from Lemmas 3.1 and 2.1. □

THEOREM 3.3.

$$\Psi_m = -xy \cdot \frac{\prod_{j=1}^m \{1 + (y-z)\, uy_j\} - \prod_{j=1}^m \{1 + (x-z)\, uy_j\}}{x \prod_{j=1}^m \{1 + (y-z)\, uy_j\} - y \prod_{j=1}^m \{1 + (x-z) uy_j\}}.$$

Proof. Direct from Theorem 3.2 and Lemma 2.2. □

4. APPLICATIONS

COROLLARY 4.1 (1) The Smirnov problem (Smirnov *et al.* [10]). *The number of sequences of type* **b** *with distinct adjacent elements is*:

$$[\mathbf{y}^{\mathbf{b}}] \left\{ 1 - \sum_{j=1}^{m} \frac{y_j}{1+y_j} \right\}^{-1}$$

This is Carlitz [1, *Theorem* 2]. *See also Eifler et al.* [6].

(2) The Simon Newcomb problem (Riordan [9]). *The number of sequences of type* **b** *with exactly r rises is*:

$$[y^r \mathbf{y}^{\mathbf{b}}] \frac{y(1-y)}{y - \prod_{j=1}^{m} \{1 + (y-1) y_j\}} .$$

This is [1, *Theorem* 4]. *See also Dillon and Roselle*, [5, *Formulas* (3.7) *and* (6.3)]. *Carlitz et al.* [3, *Formula* (3.17)] *is obtained with* $y_j = u$.

(3) A composition problem (Carlitz [2]). *The number of sequences with exactly r rises, ℓ levels, and s falls which are compositions of N into exactly m parts is* $[u^m x^s y^r z^\ell q^N]\, \Psi_m$ *where* $y_j = q^j$, $j = 1, 2,..., m$. *This is Carlitz* [2, *Theorem* 1].

Proof. Using Theorem 3.3, (1) follows with $u = 1$, $x \to y$, $x = 1$, $z = 0$; (2) follows with $u = 1$, $x \to z$, $z = 1$; (3) follows trivially. $\square$

[1, Theorems 1 and 3] follow from Theorem 3.3 with $z = 1$ and $z = 0$ respectively. Further specializations of Theorem 3.2 and 3.3 are possible.

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References

1. L. Carlitz, Enumeration of sequences by rises and falls; a refinement of the Simon Newcomb Problem, *Duke J. Math.* **39** (1972), 267–280.
2. L. Carlitz, Enumeration of compositions by rises, falls and levels, Unpublished report.
3. L. Carlitz, D. P. Roselle, and R. A. Scoville, Permutations and sequences with repetitions by number of increases, *J. Combinatorial Theory* **1** (1966), 350–374.
4. P. Cartier and D. Foata, Problèmes combinatoires de commutation et réarrangements, *in* "Lecture Notes in Mathematics," Vol. 85, Springer–Verlag, Berlin, 1969.
5. J. F. Dillon and D. P. Roselle, Simon Newcomb's problem, *SIAM J. Appl. Math.* **17** (1969), 1086–1093.
6. L. Q. Eifler, K. B. Reid, and D. P. Roselle, Sequences with adjacent elements unequal, *Aequationes Math.* **6** (1971), 256–262.
7. P. A. MacMahon, "Combinatory Analysis," Vol. 1, Chelsea, New York, 1960.
8. M. Marcus, Inequalities for matrix functions of combinatorial interest, *SIAM J. Appl. Math.* **17** (1969), 1023–1031.
9. J. Riordan, "An Introduction to Combinatorial Analysis," Wiley, New York, 1958.
10. N. V. Smirnov, O. V. Sarmanov, and V. K. Zaharov, A local limit theorem for the number of transitions in a Markov chain and its applications (Russian), *Dokl. Akad. Nauk SSR* **167** (1966), 1238–1241. [English translation: *Soviet Math. Dokl.* **7** (1966), 563–566.]